STiMS Problem Set

December 13, 2025
Emmanuel Zheng

Problem 1: Small Amplitude Water Waves

We derived the nonlinear shallow water wave equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = 0. \tag{2}$$

(a)

Show that by using equation (1), it is possible to rewrite equation (2) as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}. \tag{3}$$

This form of the momentum balance equation is very common. The left-hand side is the inertial term in Newton's second law per unit mass. The operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{4}$$

is known as the *material derivative* and is used to calculate the rate of change of a quantity (in this case u, the momentum per unit mass) carried by a fixed set of moving material particles.

(b)

The mass and momentum balance equations, (1) and (3), can be written as

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}. \tag{6}$$

Now consider waves disturbing a fluid initially at rest by writing

$$h(x,t) = H + \eta(x,t),$$

where $|\eta| \sim A$. Use scaling arguments to compare the order of magnitude of various terms in equations (5) and (6), and show that when the waves have small amplitude (i.e. the dimensionless parameter $A/H \ll 1$), the equations can be approximated by the linearized shallow water wave equations,

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}, \tag{7}$$

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}. \tag{8}$$

You will need to introduce the wavelength λ to characterize the horizontal distance over which fields vary and the period T as the characteristic time scale.

Problem 2: Energy Balance for Water Waves

As in the previous problem, consider a water layer that is invariant in the z direction. All quantities can then be defined per unit distance in the z direction.

(a)

Water waves transport kinetic and gravitational potential energy (their sum being the mechanical energy). Derive the following expression for the total mechanical energy E of the water between two points x_1 and x_2:

$$E = E_0 + \int_{x_1}^{x_2} \left[\frac{1}{2}\rho h(x,t)u(x,t)^2 + \frac{1}{2}\rho g\eta(x,t)^2\right] dx, \tag{9}$$

$$\approx E_0 + \int_{x_1}^{x_2} \left[\frac{1}{2}\rho H u(x,t)^2 + \frac{1}{2}\rho g\eta(x,t)^2\right] dx, \tag{10}$$

where the approximation is valid for small amplitude waves (i.e. in the linearized theory) and E_0 is a constant. *Hint:* Recall that the potential energy of a point mass m is mgy, where zero potential energy is defined at $y = 0$.

(b)

Using the linearized equations (7) and (8) and the associated energy (10), show that the energy balance takes the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{W}(x_1,t) - \dot{W}(x_2,t),$$

where

$$\dot{W}(x,t) = \rho g\eta(x,t)u(x,t)H \tag{11}$$

is the rate at which energy is transported in the $+x$ direction past a fixed location x, or equivalently, the rate at which fluid to the left of x does work on the fluid to the right of x. *Hint:* Start by multiplying the momentum balance by u and integrating over the control volume.

(c)

Use the expressions you have derived to show that both kinetic and potential energies per unit horizontal area scale as

$$\sim \rho g A^2.$$

Then show that

$$\dot{W} \sim \rho g A^2 c, \quad \text{where} \quad c = \sqrt{gH}.$$

It follows that energy is transported at speed c in this system.